BOUNDEDNESS OF NON-BIRATIONAL EXTREMAL CONTRACTIONS

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ABSTRACT. We consider K_X-negative extremal contractions $f \colon X \to (Z, o)$, where X is an algebraic threefold with only ϵ-log terminal $\mathbb{Q}$-factorial singularities and (Z, o) is a two (resp., one)-dimensional germ. The main result is that K_X is 1, 2, 3, 4 or 6-complementary or we have, so called, exceptional case and then the singularity $(Z \in o)$ is bounded (resp., the multiplicity of the central fiber $f^{-1}(o)$ is bounded).

INTRODUCTION

The aim of this paper is to generalize in some sense the following two statements connected with the *exceptionality* phenomenon which was discovered by Shokurov [15]. For the definition of complements we refer to 1.7.

1) The class of two-dimensional ε-log terminal singularities which have no 1 or 2-complements is bounded [15], see also [9].
2) Numerical invariants of any three-dimensional ε-log terminal singularity which have no 1, 2, 3, 4 or 6-complements are bounded [16], see also [9] for the case of quotient singularities.

There are generalizations of these facts for cases of two-dimensional elliptic fibrations (see [16]) and three-dimensional birational contractions [16]. In this paper we apply Shokurov's inductive approach to the study of contractions of relative dimension one or two. Such contractions naturally appear at the end of log MMP for varieties of Kodaira dimension $\kappa = -\infty$ (see [6]). Our main result is the following

Theorem A. *Let X be an algebraic threefold with only ε-log terminal $\mathbb{Q}$-factorial singularities and let $f \colon X \to Z$ be a K_X-negative extremal contraction (that is a projective morphism such that $f_* \mathcal{O}_X = \mathcal{O}_Z$, $\rho(X/Z) = 1$ and $-K_X$ is f-ample). Assume that $\dim Z = 2$ or 1. Fix a point $o \in Z$. Then one of the following holds.*

(i) *There exists an 1, 2, 3, 4 or 6-complement of K_X near $f^{-1}(o)$. Moreover, there exists such a complement which is not plt and even non-exceptional in the sense of 1.1.*

(ii) *K_X is exceptionally n-complementary near $f^{-1}(o)$ for $n < \mathcal{N}_2$, where $\mathcal{N}_2$ is a constant which does not depend on f. Then we have*

    (a) *in the case $\dim Z = 2$ the singularity $(Z \in o)$ is bounded, i. e. up to analytic isomorphisms it belongs to a finite set $\mathcal{M}(\varepsilon)$ depending only on ε;*

    (b) *in the case $\dim Z = 1$ the multiplicity of the central fiber $f^{-1}(o)$ is bounded by a constant $\mathcal{A}(\varepsilon)$ depending only on ε.*

For the first time the notion of complements was introduced by Shokurov in [15]. Roughly speaking an n-complement of K_X is a "good" divisor in the multiple anticanonical linear system $|-nK_X|$, see 1.7. The set of numbers $\{1, 2, 3, 4, 6\}$ from (i) of Theorem A is well known in the theory of algebraic surfaces, these numbers are called *regular*.

Following Shokurov we call case (ii) of Theorem A *exceptional*. More precisely, a contraction $f\colon X \to Z$ is said to be *exceptional* near $o \in Z$ if for any complement $K_X + D$ on X which is not Kawamata log terminal, there exists exactly one divisor S of $K(X)$ with discrepancy $a(S, D) = -1$. In this case we also show in Proposition 4.1 that the divisor S does not depend on the choice of D, so it is, in some sense, a distinguished divisor of $K(X)$. Thus Theorem A states that either K_X has a regular non-exceptional complement (see 1.1) near the fiber over $o \in Z$ or $f\colon X \to Z$ is exceptional and then we have case (ii). Shokurov in [16, §7] investigated from this point of view the case when f is a birational contraction.

There is a two-dimensional analog of Theorem A:

Proposition ([11]). *Let X be an algebraic surface having only quotient singularities and let $f\colon X \to Z$ be a K_X-negative extremal contraction onto a curve. Fix a point $o \in Z$. Then one of the following holds.*

(i) *There exists a non-exceptional 1 or 2-complement of K_X near $f^{-1}(o)$.*

(ii) *K_X is exceptionally 3-complementary near $f^{-1}(o)$. In this case X has exactly two singular points on $f^{-1}(o)$ which are of types $\frac{1}{2b-3}(1, b-2)$, $b \geq 2$ and*

$$
\begin{array}{ccccccc}
 & & & & \overset{-3}{\circ} & & \\
 & & & & | & & \\
\overset{-2}{\circ} & - & \overset{-2}{\circ} & - & \overset{-b}{\circ} & - & \overset{-2}{\circ}
\end{array}
$$

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We hope that similar to the proposition above the exceptional cases in (ii) of Theorem A can be classified.

The most important case for applications is when $\varepsilon = 1$, i. e. when X has only terminal singularities. In this case we expect that K_X is 1-complementary (this is a weak form of Reid's "general elephant" conjecture). Assuming that this conjecture is true in the case $\dim Z = 2$ we have a rough classification of contractions $f \colon X \to Z$ [10]. In particular, the base surface S has only Du Val singularities of type A_n (Iskovskikh's conjecture, [4], [10]). These conjectures have applications to the rationality problem of conic bundles (see [4]).

The idea of the proof of Theorem A is an application of Shokurov's theorem on boundedness of two-dimensional complements. We construct some special, so called plt blow-up of X (see [12]) and show that, in some sense, the geometry of exceptional divisor reflects the geometry of X itself.

The paper is organized as follows: Sect. 1 is auxiliary. In Sect. 2 we construct a new birational model of X so that we reduce the problem to dimension 2. In Sect. 3 we prove Theorem A. Examples in Sect. 4 show that boundedness as in Theorem A under weaker restrictions cannot be expected. Some of the results were announced in [13].

All varieties are assumed to be algebraic and defined over $\mathbb{C}$, the field of complex numbers. A *contraction* (or *extraction*, if we start with X instead of Y) is a projective morphism of normal varieties $f \colon Y \to X$ such that $f_* \mathcal{O}_Y = \mathcal{O}_X$. A *blow-up* is a birational extraction.

1. Preliminary results

All the necessary facts and definitions from log MMP can be found in [6], [15], [8] and [17]. We follow essentially the terminology and notation of [8], [15] and [17] (see also [7] for a nice introduction to singularities of pairs).

Definition 1.1. Let X be a normal algebraic variety and let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X. D is called a *boundary* (resp. *subboundary*) if $0 \le d_i \le 1$ (resp. $d_i \le 1$) for all i. Let $f \colon Y \to X$ be a projective birational morphism. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier and write

$$K_Y \equiv f^*(K_X + D) + \sum_E a(E, D) E,$$

where E runs over prime divisors on Y, $a(E, D) \in \mathbb{Q}$, and $a(D_i, D) = -d_i$ for each component D_i of D. The coefficients $a(E, D)$ are called

discrepancies of (X, D). Define

$$\operatorname{discr}(X, D) := \inf_{E, f}\{a(E, D) : E \text{ is } f\text{-exceptional divisor}\}.$$

Then the pair (X, D) or, by abuse of language, the divisor $K_X + D$ is said to be

Kawamata log terminal (klt)	iff $\operatorname{discr}(X, D) > -1$ and $\lfloor D \rfloor \le 0$,
purely log terminal (plt)	iff $\operatorname{discr}(X, D) > -1$,
ε-log terminal (ε-lt)	iff $\operatorname{discr}(X, D) > -1 + \varepsilon$,
log canonical (lc)	iff $\operatorname{discr}(X, D) \ge -1$.

A pair (X, D) is said to be *divisorial log terminal (dlt)* if there exists a good resolution $f\colon Y \to X$ such that the exceptional locus consists of divisors with $a(E, D) > -1$.

A log canonical pair (X, D) is said to be *exceptional* if there exists *exactly one* divisor E with discrepancy $a(E, D) = -1$. So if (X, D) is exceptional, then either (X, D) is plt and $\lfloor D \rfloor = E$ or $\lfloor D \rfloor = \emptyset$ and E is an unique exceptional divisor with $a(E, D) = -1$.

Everywhere below if we do not specify the opposite we consider log pairs (X, D) consisting of a normal variety X and a boundary D on it.

Definition 1.2 ([15, §3], [8, Ch. 16]). Let X be a normal variety, $S \ne \emptyset$ be an effective reduced divisor on X, and let B be a $\mathbb{Q}$-divisor on X, such that S, B have no common components. Assume that $K_X + S$ is lc in codimension two. Then the *different* of B on S is defined by

$$K_S + \operatorname{Diff}_S(B) \equiv (K_X + S + B)|_S.$$

Usually we will write simply Diff_S instead of $\operatorname{Diff}_S(0)$

Theorem 1.3 (Inversion of Adjunction [15, 3.3], [8, 17.6]). *Let X be a normal variety and let D be a boundary on it. Write $D = S + B$, where $S = \lfloor D \rfloor$ and $B = \{D\}$. Assume that $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then $(X, S + B)$ is plt near S iff S is normal and $(S, \operatorname{Diff}_S(B))$ is klt.*

Definition 1.4 ([12]). Let X be a normal variety and let $g\colon Y \to X$ be a blow-up such that the exceptional locus of g contains only one irreducible divisor, say S. Assume that $K_Y + S$ is plt and $-(K_Y + S)$ is f-ample. Then $g\colon (Y \supset S) \to X$ is called a *purely log terminal (plt) blow-up* of X.

Definition 1.5 ([15, 3.14]). Let X be a normal variety and let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety $W \subset X$ is said to be a *center of log canonical singularities* for (X, D) if there exists a prime (not necessary exceptional) divisor E over

X with center at W and discrepancy $a(E, D) \leq -1$. The union of all centers of log canonical singularities is called the *locus of log canonical singularities* and denoted by $LCS(X, D)$.

For the following statement we refer to [8, 17.4] (in dimension 2 it was proved earlier by Shokurov [15]).

Theorem 1.6 (Connectedness Lemma). *Let X be a normal projective variety, let $f \colon X \to Z$ be a contraction and let $D = \sum d_i D_i$ be a boundary on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $-(K_X + D)$ is f-nef and f-big, then $LCS(X, D)$ is connected near each fiber of f.*

V. V. Shokurov informed me that the result above has a generalization modulo log MMP to the case when $-(K_X + D)$ is only nef (preprint, in preparation, cf. [15, 6.9][1]). We need only a particular case of this fact (see 2.16).

Definition 1.7 ([15, 5.1]). Let X be a normal variety and let $D = S + B$ be a subboundary, such that B, S have no common components, S is a reduced divisor, and $\lfloor B \rfloor \leq 0$. Then one says that $K_X + D$ is *n-complementary*, if there exists a $\mathbb{Q}$-divisor D^+ such that

(i) nD^+ is integer and $n(K_X + D^+) \sim 0$;
(ii) $K_X + D^+$ is lc;
(iii) $nD^+ \geq nS + \lfloor (n+1)B \rfloor$.

The divisor $K_X + D^+$ is called an *n-complement* of $K_X + D$.

Remark 1.8. Note that in general it is not true that $D^+ \geq D$. This however is true if all the coefficients of D are *standard*, i. e. they have the form $d_i = 1 - 1/m_i$, where $m_i \in \mathbb{N} \cup \{\infty\}$ [16, 2.7].

For convenience, we recall several facts about complements. For proofs we refer to [15], [8], [16] and author's lecture notes [14].

Lemma 1.9 ([15, 5.4]). *Let $f \colon Y \to X$ be a blow-up and let D be a subboundary on Y. Assume that $K_Y + D$ is n-complementary. Then $K_X + f_* D$ is n-complementary.*

Lemma 1.10 ([16, 4.4]). *Let $f \colon Y \to X$ be a blow-up and let D be a subboundary on Y such that*

(i) $K_Y + D$ *is f-nef;*
(ii) $f_* D = \sum d_i f_* D_i$ *is a boundary with standard coefficients.*

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[1] Recently a generalization of this fact was obtained by O. Fujino [3, Proposition 2.1]

Assume that $K_X + f_*D$ is n-complementary and let $K_X + (f_*D)^+$ be any n-complement. Then its crepant pull-back $K_X + (f_*D)^+$ is an n-complement of $K_Y + D$.

Lemma 1.11 ([11, 2.1], cf. [8, 19.6]). *Let (X, S) be a plt pair with reduced $S \neq 0$. Let $f\colon X \to Z$ be a contraction such that $-(K_X + S)$ is f-nef and f-big. Fix a fiber $f^{-1}(o)$, $o \in Z$ meeting S. Assume that $K_S + \mathrm{Diff}_S$ is n-complementary. Then any n-complement of $K_S + \mathrm{Diff}_S$ in a neighborhood of $f^{-1}(o)$ can be extended to an n-complement of $K_X + S$. This means that for any n-complement $K_S + \mathrm{Diff}_S^+$ there exists an n-complement $K_X + S + B^+$ such that $\mathrm{Diff}_S(B^+) = \mathrm{Diff}_S^+$.*

Proof. Let $K_S + \Theta$ be an n-complement of $K_S + \mathrm{Diff}_S$. Consider a good resolution $g\colon Y \to X$ and put $h = f \circ g$. Write

$$g^*(K_X + S) = K_Y + S_Y + B,$$

where S_Y is the proper transform of S and B is a subboundary on Y such that $\lfloor B \rfloor \leq 0$. Then $K_Y + S_Y + B$ is plt [7, 3.10]. By Inversion of Adjunction $g^*(K_S + \mathrm{Diff}_S) = K_{S_Y} + \mathrm{Diff}_{S_Y}(B)$ is klt and since Y is non-singular, $\mathrm{Diff}_{S_Y}(B) = B|_{S_Y}$. By Lemma 1.10 $K_{S_Y} + \Theta_Y :=$ $g^*(K_S + \Theta)$ is an n-complement of $K_{S_Y} + \mathrm{Diff}_{S_Y}(B)$. In particular, $\overline{\Theta_Y} := n\Theta_Y - \lfloor (n+1)\mathrm{Diff}_{S_Y}(B) \rfloor$ is an effective divisor from the linear system $| - nK_{S_Y} - \lfloor (n+1)B \rfloor |_{S_Y}|$ (see (iii) of 1.7).

From the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)B \rfloor)$$
$$\longrightarrow \mathcal{O}_Y(-nK_Y - nS_Y - \lfloor (n+1)B \rfloor)$$
$$\longrightarrow \mathcal{O}_{S_Y}(-nK_{S_Y} - \lfloor (n+1)B \rfloor |_{S_Y}) \longrightarrow 0$$

and the vanishing

$$R^1 h_*(\mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)B \rfloor)) =$$
$$R^1 h_*(\mathcal{O}_Y(K_Y + \lceil -(n+1)(K_Y + S_Y + B) \rceil)) = 0.$$

We get surjectivity of the restriction

$$H^0(Y, \mathcal{O}_Y(-nK_Y - nS_Y - \lfloor (n+1)B \rfloor)) \longrightarrow$$
$$H^0(S_Y, \mathcal{O}_{S_Y}(-nK_{S_Y} - \lfloor (n+1)B \rfloor |_{S_Y})).$$

Therefore there is an element $\overline{D} \in | - nK_Y - nS_Y - \lfloor (n+1)B \rfloor |$ such that $\overline{D}|_{S_Y} = \overline{\Theta_Y}$. Put $B_Y^+ := \frac{1}{n}(\lfloor (n+1)B \rfloor + \overline{D})$ and $B^+ := g_* B_Y^+$. Then we have $n(K_Y + S_Y + B_Y^+) \sim 0$ and $(K_Y + S_Y + B_Y^+)|_{S_Y} = K_{S_Y} + \Theta_Y$. This gives us $n(K_X + S + B^+) \sim 0$ and $(K_X + S + B^+)|_S = K_S + \Theta$. By our construction B^+ is effective, so it is a boundary. Now we have to

show only that $K_X + S + B^+$ is lc. Since $K_S + \text{Diff}_S$ is klt and $K_S + \Theta$ is lc, we have that $K_S + \text{Diff}_S + \alpha(\Theta - \text{Diff}_S = (K_X + S + \alpha B^+)|_S$ is klt for $\alpha < 1$. By Inversion of Adjunction $K_X + S + \alpha B^+$ is plt near S for $\alpha < 1$. Therefore $K_X + S + B^+$ is lc near S. Moreover, by Connectedness Lemma $LCS(X, S + \alpha B^+)$ is connected near each fiber of f for $0 < \alpha < 1$ (because $-(K_X + S + \alpha B^+)$ is f-nef and f-big). Therefore $LCS(X, S + \alpha B^+) = S$ and $K_X + S + \alpha B^+$ is plt for $\alpha < 1$. This gives us that $K_X + S + B^+$ is lc and proves the lemma. □

Theorem 1.12 ([16]). *Let S be a projective surface and let Δ be a boundary on it. Assume that*

(i) *coefficients of Δ are standard, i. e. $\Delta = \sum(1 - 1/m_i)\Delta_i$, $m_i \in \mathbb{N} \cup \{\infty\}$;*
(ii) *$K_S + \Delta$ is lc;*
(iii) *$-(K_S + \Delta)$ is nef and big.*

Then there exists an n-complement of $K_S + \Delta$ such that $n < \mathcal{N}_2$, where $\mathcal{N}_2$ is an absolute constant which does not depend on (S, Δ).

2. Construction of a good model

In this section we modify the techniques developed in [15, §6], [8, §21] to our situation. Roughly speaking we have to construct a "good" model for a plt blow-up of our threefold X from Theorem A in order to apply Lemma 1.11 and Theorem 1.12. A similar techniques was applied by Shokurov in [16, §7] for the study of the birational case. However we employ here a little different method which allow us to work with irreducible exceptional divisor. This construction gives us the existence of an n-complement on X which is exceptional and $n < \mathcal{N}_2$.

2.1. **Notation.** Let X be an algebraic threefold with only klt $\mathbb{Q}$-factorial singularities, $f \colon X \to Z$ be an extremal contraction on a (normal) variety Z of positive dimension, i. e. we assume that $\rho(X/Z) = 1$ and $-K_X$ is f-ample. Fix a point $o \in Z$ and assume that Z is a germ near o. In this section we do not assume that $\dim Z \neq 3$. Thus we consider the following cases: 1) del Pezzo fibrations, 2) generically conic bundle fibrations, 3) birational contractions.

2.2. Take a boundary F on X such that

(i) $-(K_X + F)$ is f-ample;
(ii) $K_X + F$ is lc but not klt near $f^{-1}(o)$.

By our assumptions, $\dim Z \geq 1$, so we can take simply $F = cf^*L$, where L is an effective divisor on Z containing o and c is the *log canonical threshold* of (X, f^*L) that is maximal $c \in \mathbb{Q}$ such that $K_X + cf^*L$ is lc. Of course there are another possibilities for the choice of F.

First we consider the easy case, when $K_X + F$ is plt. It is clear that $\lfloor F \rfloor \neq \emptyset$, because $K_X + F$ is not klt. If $\dim Z = 2$ and under the assumption that X has only terminal singularities this case was studied in [11].

Lemma 2.3 (cf. [15, 5.12]). *Notation as above. Assume that $K_X + F$ is plt (near $f^{-1}(o)$). Then*

(i) $K_X + \lfloor F \rfloor$ *is n-complementary for $n < \mathcal{N}_2$;*

(ii) *if $\lfloor F \rfloor$ contains a non-compact component, then $K_X + \lfloor F \rfloor$ has a regular complement which is not plt.*

Note that by Connectedness Lemma 1.6 $\lfloor F \rfloor$ is connected near $f^{-1}(o)$, so it is irreducible. Moreover, if $\lfloor F \rfloor$ is compact, then $\lfloor F \rfloor = f^{-1}(o)$. This is impossible if $\dim Z = 2$, because f is equidimensional.

Proof. First we assume that $-(K_X + \lfloor F \rfloor)$ is f-ample and so is $-(K_{\lfloor F \rfloor} + \mathrm{Diff}_{\lfloor F \rfloor})$. By Adjunction 1.3 $K_{\lfloor F \rfloor} + \mathrm{Diff}_{\lfloor F \rfloor}$ is klt and $\mathrm{Diff}_{\lfloor F \rfloor}$ has only standard coefficients [15, 3.9], [8, 16.6]. By Theorem 1.12 $K_{\lfloor F \rfloor} + \mathrm{Diff}_{\lfloor F \rfloor}$ is n-complementary for some $n < \mathcal{N}_2$. Moreover, we can take $n \in \{1, 2, 3, 4, 6\}$ if $\lfloor F \rfloor$ is non-compact, see [15, 5.6] for $\dim f(\lfloor F \rfloor) = 2$ and [16, §3] for $\dim f(\lfloor F \rfloor) = 1$ (see also [8] and [14]). Since $-(K_X + \lfloor F \rfloor)$ is f-ample, by Lemma 1.11 complements of $K_{\lfloor F \rfloor} + \mathrm{Diff}_{\lfloor F \rfloor}$ can be extended to complements of $K_X + \lfloor F \rfloor$ near the fiber over $o \in Z$.

Now we assume that $-(K_X + \lfloor F \rfloor)$ is not f-ample. Then $K_X + \lfloor F \rfloor$ is f-nef, because $\rho(X/Z) = 1$. Therefore $-\{F\}$ is f-ample. This is possible only if f is birational. If f is divisorial, then Z is $\mathbb{Q}$-factorial and $K_Z + f_*F$ is plt, hence so is $K_Z + f_*\lfloor F \rfloor$. Moreover, $\dim f(\lfloor F \rfloor) = 2$, because f contracts a component of $\mathrm{Supp}\{F\}$. Hence $f_*\lfloor F \rfloor$ is non-compact. As above, $K_{f_*\lfloor F \rfloor} + \mathrm{Diff}_{f_*\lfloor F \rfloor}$ has a regular complement. By Lemma 1.11 this complement can be extended on Z near o and by Lemma 1.10 there exists a regular complement of $K_X + \lfloor F \rfloor$ near $f^{-1}(o)$.

Finally, assume that f is small. Then there exists a $(K_X + F)$-flip $X \dashrightarrow X^+$. On X^+ the proper transform $K_{X^+} + \lfloor F \rfloor^+$ of $K_X + \lfloor F \rfloor$ is antiample or numerically trivial over Z. Again by Lemma 1.11 there is a regular complement on X^+ and taking pull-back it on X we get the desired complement. $\square$

Now we consider the case when $K_X + F$ is not plt.

Lemma 2.4 ([12], cf. [15, 9.1], [8, 17.4], [17]). *Let X be a normal $\mathbb{Q}$-factorial variety of dimension ≤ 3 and let F be a boundary on X such*

that $K_X + F$ is lc, but not plt. Assume that $F \neq 0$ and X has at worse klt singularities. Then there exists a plt blow-up $g \colon Y \to X$ such that

(i) Y is $\mathbb{Q}$-factorial, $\rho(Y/X) = 1$ and the exceptional locus of g is an irreducible divisor S;

(ii) $K_Y + S + B = g^*(K_X + F)$ is lc;

(iii) $K_Y + S + (1 - \varepsilon)B$ is plt and antiample over X for any $\varepsilon > 0$.

Lemma 2.5. *Notation as in 2.1. Let F be such as in 2.2. Assume that $K_X + F$ is not plt. Then there exists a plt blow-up $g \colon (Y, S) \to X$ and a complement $K_Y + S + A$ which is plt.*

Note that complements $K_Y + S + A$ and $K_X + g_*A$ are exceptional in the sense of 1.1.

Proof. Apply Lemma 2.4 to (X, F). We get a plt blow-up $g \colon Y \to X$ with log canonical $g^*(K_X + F) = K_Y + S + B$. Since $\rho(Y/Z) = 2$, the cone $\overline{NE}(Y/Z)$ is generated by two extremal rays. Denote them by R and Q. One of them, say R, determines the contraction $g \colon Y \to X$ and therefore it is trivial with respect to $K_Y + S + B$, so B must be positive on R. Since $K_Y + S + B = g^*(K_X + F)$, where $K_X + F$ antiample over Z, $K_Y + S + B$ cannot be nef. Therefore $K_Y + S + B$ is negative on Q. Take small $\beta > 0$ and denote $B^\beta := (1 - \beta)B$.

Then $K_Y + S + B^\beta$ is plt and antiample over Z. For sufficiently big and divisible $n \in \mathbb{N}$ the divisor $-n(K_Y + S + B^\beta)$ is very ample over Z. Take a general member $L \in |-n(K_Y + S + B^\beta)|$ and denote $A := B^\beta + (1/n)L$. Then $K_Y + S + A$ is an n-complement of $K_Y + S$. Moreover, $K_Y + S + A$ is plt, see [7, 4.7]. $\qquad\square$
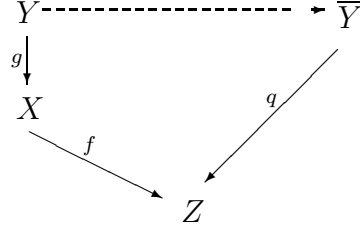
Remark 2.6. Since $K_Y + S + B^\beta$ is plt and antiample over Z, $R^1(g \circ f)_*\mathcal{O}_Y = 0$. This gives us that the numerical equivalence of (Cartier) divisors on Y coincides with $\mathbb{Q}$-linear one.

2.7. Now let $f \colon X \to Z$ be a contraction as in 2.1, let $g \colon (Y, S) \to X$ be a plt blow-up from Lemma 2.5 and let $K_Y + S + A$ be a complement on Y which is plt.

Note that $-(K_Y + S)$ is not necessarily nef over Z (so we cannot apply Lemma 1.11 directly). To improve the situation we consider the following construction.
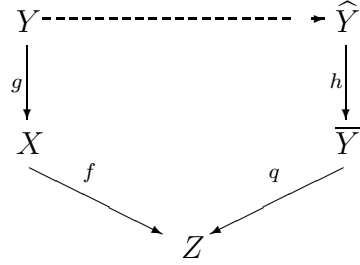
Proposition 2.8. *Notation as in 2.7. One of the following holds.*

(A) *After a sequence of $(K_Y + S + (1+\varepsilon)A)$-flips $Y \dashrightarrow \overline{Y}$ over $Z \ni o$, we get a log variety $(\overline{Y}, \overline{S} + (1+\varepsilon)\overline{A})$ and the diagram*

such that $K_{\overline{Y}} + \overline{S}$ is plt and $-(K_{\overline{Y}} + \overline{S})$ is q-nef and q-big.

(B) *After a sequence of $(K_Y + S + (1+\varepsilon)A)$-flips $Y \dashrightarrow \widehat{Y}$ over $Z \ni o$, we get a log variety $(\widehat{Y}, \widehat{S} + (1+\varepsilon)\widehat{A})$ and the diagram*

where $h \colon \widehat{Y} \to \overline{X}$ is a divisorial contraction which is positive with respect to $K_{\widehat{Y}} + \widehat{S} \equiv -\widehat{A}$ and negative with respect to $K_{\widehat{Y}} + \widehat{S} + (1+\varepsilon)\widehat{A}$. In this case $\overline{S} := h(\widehat{S})$ is a surface, $K_{\overline{Y}} + \overline{S}$ is plt and $-(K_{\overline{Y}} + \overline{S})$ is ample over Z.

Proof. For sufficiently small $\varepsilon > 0$ the divisor $K_Y + S + (1+\varepsilon)A$ is also plt (see [15, 1.3.4], [8, 2.17.4]). Apply $(K_Y + S + (1+\varepsilon)A)$-MMP to Y. After a number of flips we get one of the following:

- A log pair $(\overline{Y}, \overline{S} + (1+\varepsilon)\overline{A})$, where $K_{\overline{Y}} + \overline{S} + (1+\varepsilon)\overline{A}$ is nef over Z. Since $K_{\overline{Y}} + \overline{S} + \overline{A} \equiv 0$, so are both $\overline{A}$ and $-(K_{\overline{Y}} + \overline{S})$. This is case (A) of 2.8.
- A log pair $(\widehat{Y}, \widehat{S} + (1+\varepsilon)\widehat{A})$ with a non-flipping extremal contraction $h \colon \widehat{Y} \to \overline{Y}$ which is negative with respect to $K_{\widehat{Y}} + \widehat{S} + (1+\varepsilon)\widehat{A}$. This contraction must be negative with respect to $\widehat{A}$, because $K_{\widehat{Y}} + \widehat{S} + \widehat{A} \equiv 0$ over Z. Therefore it is divisorial and contracts a component of $\operatorname{Supp}(\widehat{A})$. In particular, $\overline{S} := h(\widehat{S})$ is a divisor. Put $\overline{A} := \widehat{A}$. Then $-(K_{\overline{Y}} + \overline{S}) \sim_{\mathbb{Q}} \overline{A})$ is big over Z. By Kodaira's Lemma (see e. g. [6, 0-3-4]),

$$-(K_{\overline{Y}} + \overline{S}) \sim_{\mathbb{Q}} (q\text{-ample}) + (\text{effective}).$$

If q is not birational, then $-(K_{\overline{Y}} + \overline{S})$ is q-ample because $\rho(\overline{Y}/Z) = 1$. If q is birational, then it contracts $\overline{S}$. This gives us that $f \circ g$

contracts two divisors. Hence f is a divisorial contraction and $Z \ni o$ is klt and $\mathbb{Q}$-factorial [6, 5-1-6]. Then $-(K_{\overline{Y}} + \overline{S})$ must be q-ample. We get case (B) of 2.8.

$\square$

Remark 2.9. By 2.6 the numerical equivalence on Y coincides with $\mathbb{Q}$-linear one. All flips and the divisorial contraction $\widehat{Y} \dashrightarrow \overline{Y}$ preserve this property, so the same holds on $\overline{Y}$.

Proposition 2.10. *In notation 2.1 assume that $f(g(S))$ is a curve. Then K_X has a regular non-exceptional complement.*

Proof. Consider case (A) of Proposition 2.8. It is easy to see that $q(\overline{S}) = f(g(S))$ is also a curve. Denote this curve by C. We have a contraction $q \colon \overline{S} \to C \ni o$. It is clear that $-(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}) = -(K_{\overline{Y}} + \overline{S})|_{\overline{S}}$ is nef over C. By [16, 3.1] there exists a regular complement of $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}$ near $q^{-1}(o)$. By Lemma 1.11 this complement can be extended to a regular complement $K_{\overline{Y}} + \overline{S} + \overline{D}$ on $\overline{Y}$. The sequence of maps

$$\chi \colon \overline{Y} \dashrightarrow \cdots \dashrightarrow Y$$

is a sequence of flops with respect to $K_{\overline{Y}} + \overline{S} + \overline{D}$. They preserve the lc property of $K_{\overline{Y}} + \overline{S} + \overline{D}$ (see [8, 2.28]). This gives us a regular complement $K_Y + S + D = \chi_*(K_{\overline{Y}} + \overline{S} + \overline{D})$. Finally, $K_X + g_*D$ is a regular complement on X and $a(S, g_*D) = -1$.

   Case (B) of Proposition 2.8 can be treated by the similar way: we can construct a complement on $\overline{S}$, extend it on $\overline{Y}$, pull-back it on $\widehat{Y}$ by Lemma 1.10 and take the proper transform on X. $\square$

   Thus if $f(g(S))$ is a curve, then we have case (i) of Theorem A. From now on we assume that $f(g(S))$ is a point, so both S and $\overline{S}$ are compact surfaces.

Lemma 2.11. *Notation as in 2.7. Assume that $f(g(S))$ is a point. Then $-(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}})$ is nef and big on $\overline{S}$ (by definition, $(\overline{S}, \mathrm{Diff}_{\overline{S}})$ is a weak log del Pezzo).*

Proof. Since $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}} = (K_{\overline{Y}} + \overline{S})|_{\overline{S}}$, the divisor $-(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}})$ is nef. If $-(K_{\overline{Y}} + \overline{S})$ is ample over Z, then obviously, so is $-(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}})$. Thus we assume that $-(K_{\overline{Y}} + \overline{S})$ is not ample over Z. This is possible only in case (A) of 2.8.

   By of Proposition 2.8 $-(K_{\overline{Y}} + \overline{S})$ is nef and big. Recall that $\rho(\overline{Y}/Z) = 2$, hence $NE(\overline{Y}/Z)$ is generated by two extremal rays. If $-(K_{\overline{Y}} + \overline{S})$ is

not ample, then it is trivial on some extremal ray $\overline{Q}$ on $\overline{Y}$. We use the notation of the proof of Lemma 2.5. Let $\overline{B^\beta}$ be the proper transform of B^β.

First we consider the case $\overline{B^\beta} \cdot \overline{Q} < 0$. We claim that $\overline{Q}$ is contractible over Z. Indeed, $\overline{Q}$ is negative with respect to $K_{\overline{Y}} + \overline{S} + \varepsilon \overline{B^\beta}$, where $\varepsilon > 0$. Thus $\overline{Q}$ is contractible by Contraction Theorem [6, 3-2-1]. Let $\overline{g} \colon \overline{Y} \to W$ be the contraction of $\overline{Q}$ over Z. Then $-(K_{\overline{Y}} + \overline{S}) = \overline{g}^*(L)$, where the $\mathbb{Q}$-divisor L is ample on W, because $\rho(W/Z) = 1$. Therefore the linear system $|-n(K_{\overline{Y}} + \overline{S})|$ is base point free over Z for $n \gg 0$. Further, $\overline{B^\beta} \cdot \overline{Q} < 0$, hence the contraction is birational and contracts a component of $\mathrm{Supp}(\overline{B^\beta})$ or it is small. In both cases the morphism determined by the linear system $|-n(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}})|$ can contract only a finite number of curves in $\overline{S} \cap \mathrm{Supp}(\overline{B^\beta})$, i. e. $-(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}})$ is big.

Similarly, in the case $\overline{B^\beta} \cdot \overline{Q} > 0$ we have that $\overline{Q}$ is negative with respect to $K_{\overline{Y}} + \overline{S} + \varepsilon(\overline{A} - \overline{B^\beta})$ (recall that $\overline{A} - \overline{B^\beta}$ is effective).

Finally, if $\overline{B^\beta} \cdot \overline{Q} = 0$, then $K_{\overline{Y}} + \overline{S}$ and $\overline{B^\beta}$ are numerically proportional, because $\rho(\overline{Y}/Z) = 2$. By 2.9 $K_{\overline{Y}} + \overline{S} \sim_{\mathbb{Q}} c\overline{B^\beta}$ for some $c \in \mathbb{Q}$. Whence $K_Y + S \sim_{\mathbb{Q}} cB^\beta = c(1-\beta)B$ and $K_Y + S + B \equiv (1 + c(1-\beta))B$. Since $-(K_Y + S)$ is g-ample and $K_Y + S + B$ is g-numerically trivial, we have $1 + c(1 - \beta) = 0$. So $g^*(K_X + F) = K_Y + S + B \equiv 0$, a contradiction. This proves the lemma. $\square$

Proposition 2.12. *Let $f \colon X \to Z$ be as in 2.1. Then the canonical divisor K_X is n-complementary for some $n < \mathcal{N}_2$, where $\mathcal{N}_2$ is the constant from Theorem 1.12.*

Proof. First we consider case (A) of Proposition 2.8. By Shokurov's theorem 1.12 we have an n-complement $K_{\overline{S}} + \overline{\Theta}$ of $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}$, where $n < \mathcal{N}_2$. By Lemma 1.11 $K_{\overline{S}} + \overline{\Theta}$ can be extended to an n-complement $K_{\overline{Y}} + \overline{S} + \overline{D}$ on $\overline{Y}$. As in the proof of Proposition 2.10 its proper transform on X is an n-complement of K_X.

In case (B) of Proposition 2.8 we have a contraction $q \colon \overline{X} \to Z$ and a boundary $\overline{S} := h(\widehat{S})$ such as in Lemma 2.3. Therefore $K_{\overline{X}} + \overline{S}$ is n-complementary for $n < \mathcal{N}_2$. We can pull-back this complement on $\overline{Y}$ by Lemma 1.10. The rest is similar to case (A). $\square$

Proposition 2.13. *Notation as above. Assume that there exists a complement $K_{\overline{S}} + \overline{\Theta}$ of $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}$ which is not klt. Then K_X has a regular non-exceptional complement.*

Proof. Assume that $K_{\overline{S}} + \overline{\Theta}$ is not klt. As in the proof of Proposition 2.10 it is sufficient to show that $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}$ has a regular non-klt complement.

The following is an easy consequence of [16, 2.3] (see [12, Corollary 11]).

Lemma 2.14. *Let S be a projective normal surface and let $\Theta \geq \Delta$ be boundaries on S such that*
  (i) *coefficients of Δ are standard;*
  (ii) *$K_S + \Delta$ is klt;*
  (iii) *$-(K_S + \Delta)$ is ample;*
  (iv) *$K_S + \Theta$ is lc but not klt;*
  (v) *$-(K_S + \Theta)$ is nef.*
Then $K_S + \Delta$ has a regular complement which is not klt.

By this lemma $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}$ has a regular non-klt complement. By Lemma 1.11 it can be extended on $\overline{X}$. The proper transform of it on X gives us a regular complement which is non-exceptional. $\qquad\square$

We obtain very important

Corollary 2.15. *Let $f\colon X \to Z$ be as in 2.1. Let $g\colon (Y, S) \to X$ be a plt blow-up constructed in 2.5 and let $K_Y + S + D$ be some complement. If $K_Y + S + D$ is not plt, then $K_Y + S$ has a regular non-plt complement.*

Proof. Notation as in the proof of Proposition 2.8. We claim that the proper transform $K_{\overline{Y}} + \overline{S} + \overline{D}$ of $K_Y + S + D$ on $\overline{Y}$ is not plt. Indeed, in case (A) $Y \to \overline{Y}$ is a sequence of flops with respect to $K + S + D$ and the fact follows by [8, 2.28]. Similarly, in case (B) $K_{\widehat{Y}} + \widehat{S} + \widehat{D}$ is not plt. The contraction $h\colon \widehat{Y} \to \overline{X}$ is log crepant with respect to $K_{\widehat{Y}} + \widehat{S} + \widehat{D}$, so $K_{\overline{Y}} + \overline{S} + \overline{D}$ is not plt by [7, 3.10].

By Lemma 2.16 below $K_{\overline{Y}} + \overline{S} + \overline{D}$ is not plt near $\overline{S}$. Therefore $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\overline{D})$ is not klt. Now the corollary follows by Proposition 2.13. $\quad\square$

Lemma 2.16. *Let $f\colon X \to Z$ be a contraction from a threefold with $\dim Z > 0$ and let D be a boundary on X. Fix a point $o \in Z$. Assume that*
  (i) *$K_X + D$ is lc and not plt near $f^{-1}(o)$;*
  (ii) *$K_X + D \equiv 0$ over Z;*
  (iii) *there is an irreducible component $S_0 \subset \lfloor D \rfloor$ such that $f(S_0) \neq Z$.*
Then $K_X + D$ is not plt near $S_0 \cap f^{-1}(o)$.

In the proof we follow the proof of [15, 6.9][2].

---

[2]The referee pointed out that Lemma 2.16 easily follows from [3, Proposition 2.1]

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Proof. If f is birational, then the lemma follows by Theorem 1.6. We assume that Z is one or two-dimensional. Consider Z as a sufficiently small neighborhood of o. Replace (X, D) with its $\mathbb{Q}$-factorial dlt model (see [15, 9.1] or [8, 8.2.2]). Put $B := \{D\}$ and $S := \lfloor D \rfloor$. Since $K_X + S + B$ is not plt, S has at least two irreducible components. Assume that $K_X + S + B$ is plt near S_0. Then there are no irreducible components $S_i \neq S_0$ of S meeting S_0. Hence S_0 is a connected component of S. Thus we have

(i)$'$ $K_X + S + B$ is lc over the germ $o \in Z$ and X is $\mathbb{Q}$-factorial;

(ii)$'$ $K_X + S + B \equiv 0$ over Z;

(iii)$'$ S_0 is a connected component of S, it is irreducible and $f(S_0) \neq Z$;

(iv)$'$ $LCS(X, S + B) = S$ and $S \neq S_0$.

If $K_X + B \equiv -S$ is nef, then $S \supset f^{-1}(o)$. Indeed, otherwise there is a curve $C \subset f^{-1}(o)$ such that $C \cdot S > 0$. Obviously, in this case S is connected near $f^{-1}(o)$, a contradiction. Now we assume that $K_X + B$ is not nef. Run $(K_X + B)$-MMP over Z. On each step $K_X + B$ is klt. Hence $K_X + S + B$ is klt outside of S and $LCS(X, S + B) = S$. Let R be a $(K_X + B)$-negative extremal ray and let $\psi \colon X \to X'$ be its contraction. Since $R \cdot S > 0$, ψ cannot contract a connected component of S. In particular, ψ cannot contract S_0. Further, by Theorem 1.6 S is connected near each fiber of ψ. Therefore the number of connected components of S remains the same. We have shown that properties (i)$'$–(iv)$'$ hold on every step of MMP.

At the end we get a non-birational $(K_X + B)$-negative extremal contraction $f' \colon X \to Z'$ over Z. Then $S \equiv -(K_X + B)$ is f'-ample. On the other hand, $f'(S_0) \neq Z'$, so $S_0 \equiv 0$ over Z'. Write $S = \sum S_i + \sum S_j$, where S_i's are f'-ample and S_j's are f'-numerically trivial components. By the above remarks, both terms are non-empty. Any component of type S_i intersects all curves in fibers of f'. In particular, it intersects any component of type S_j. Therefore S is connected. This contradicts to our assumptions (iii)$'$–(iv)$'$. $\square$

Definition 2.17. Let $f \colon X \to Z \ni o$ be a contraction such as in 2.1. Then it is said to be *exceptional* if any non-klt complement $K_X + D$ near $f^{-1}(o)$ is exceptional.

Corollary 2.18. *Let $f \colon X \to Z$ be as in 2.1 and assume that it is not exceptional. Then K_X has a regular complement which is also non-exceptional.*

3. PROOF OF THEOREM A

In this section we prove Theorem A.

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Proposition 3.1. *Notation as in 2.1 and 2.2. Assume that $K_X + F$ is plt. Let $\psi \colon X' \to X$ be a finite étale in codimension one cover. Assume that all complements of $K_X + \lfloor F \rfloor$ are exceptional (in this case it means that they are plt). Then $\deg \psi$ is bounded by an absolute constant.*

In case when $K_X + F$ is plt this assertion gives us a stronger result than (ii) of Theorem A because here the absolute constant does not depend on ε.

Proof. By Lemma 2.3 we may assume that $\lfloor F \rfloor$ is compact and $-(K_X + \lfloor F \rfloor)$ is f-ample. Thus $\lfloor F \rfloor = f^{-1}(o)$. Denote $\Delta := \mathrm{Diff}_{\lfloor F \rfloor}$, $m := \deg \psi$. If $K_{\lfloor F \rfloor} + \Delta$ has a regular complement, then by Lemma 1.11 it can be extended on X, so we assume the opposite. By Theorem 1.12 $K_{\lfloor F \rfloor} + \Delta$ has an n-complement $K_{\lfloor F \rfloor} + \Theta$ for $n < \mathcal{N}_2$. Moreover, we may assume that $K_{\lfloor F \rfloor} + \Theta$ is klt (see Lemma 2.14). Therefore both $K_{\lfloor F \rfloor} + \Theta$ and $K_{\lfloor F \rfloor} + \Delta$ are $(1/\mathcal{N}_2)$-lt, because $n(K_{\lfloor F \rfloor} + \Theta) \sim 0$. Now put $F' := \psi^{-1} F$. It is easy to see $\lfloor F' \rfloor = \psi^{-1}(\lfloor F \rfloor)$. Since $\psi \colon X' \to X$ is étale in codimension one,

$$K_{X'} + F' = \psi^*(K_X + F), \qquad K_{X'} + \lfloor F' \rfloor = \psi^*(K_X + \lfloor F \rfloor).$$

By [15, §2] (see also [8, 20.3], [7, 3.16]) we have that $K_{X'} + F'$ is plt. So, by Inversion of Adjunction and by Connectedness Lemma $\lfloor F' \rfloor$ is irreducible and normal. It is clear that the restriction $\varphi = \psi|_{\lfloor F' \rfloor} \colon \lfloor F' \rfloor \to \lfloor F \rfloor$ is a finite morphism of the same degree as ψ. Denote $\Delta' := \mathrm{Diff}_{\lfloor F' \rfloor}$. Thus by Adjunction 1.3 we have $K_{\lfloor F' \rfloor} + \Delta' = \varphi^*(K_{\lfloor F \rfloor} + \Delta)$. Therefore $(\lfloor F' \rfloor, \Delta')$ is also $(1/\mathcal{N}_2)$-lt (see [15, Sect. 2] and also [8, 20.3]). Similarly, $K_{\lfloor F' \rfloor} + \Theta' = \varphi^*(K_{\lfloor F \rfloor} + \Theta)$, where $K_{\lfloor F' \rfloor} + \Theta'$ is an n-complement of $K_{\lfloor F' \rfloor} + \Delta'$. By [2] the surface $\lfloor F' \rfloor$ belongs to a finite number of families. Since $\Theta' \in |-nK_{\lfloor F' \rfloor}|$, the set $\{(\lfloor F' \rfloor, \Theta')\}$ is also bounded. Taking into account $\Delta' \le \Theta'$, we have that so is $\{(\lfloor F' \rfloor, \Delta')\}$. Therefore $0 < (K_{\lfloor F' \rfloor} + \Delta')^2 < \mathrm{Const}$. Finally, the equality $(K_{\lfloor F' \rfloor} + \Delta')^2 = (\deg \varphi)(K_{\lfloor F \rfloor} + \Delta)^2$ gives us that $\deg \varphi = \deg \psi$ is bounded and proves the proposition. $\qquad\square$

The following is the main step in the proof of Theorem A.

Proposition 3.2. *Let $f \colon X \to Z \ni o$ be an contraction as in 2.1. Assume that X has only ε-lt singularities, where $\varepsilon > 0$. Assume also that $f \colon X \to Z \ni o$ is exceptional. Let $\psi \colon X' \to X$ be a finite Galois étale in codimension one cover with connected X'. Then the degree of $\psi \colon X' \to X$ is bounded by a constant depending only on ε.*

Proof. First note that since X is a germ along $f^{-1}(o)$, $(f \circ \psi)^{-1}(o)$ is connected.

Below we will use notation of 2.7. Taking into account Proposition 3.1 we may assume that $K_X + F$ is not plt. Let $g\colon (Y, S) \to X$ be a plt blow-up from Lemma 2.5. By Proposition 2.10 we may assume that $f \circ g(S)$ is a point and By Proposition 2.12 there exists an n-complement $K_Y + S + D$, where $n < \mathcal{N}_2$. Denote $\Delta := \mathrm{Diff}_S$ and $\Theta := \mathrm{Diff}_S(D)$. By our assumptions and by 2.15 $K_Y + S + D$ is plt hence $K_S + \Theta$ is klt. Since $n(K_S + \Theta) \sim 0$, $K_S + \Theta$ is also $(1/\mathcal{N}_2)$-lt. As in the proof of Proposition 3.1 by [2] S lies in a finite number of families and because $n\Theta \in |-nK_S|$, the pair (S, Θ) also lies in a finite number of families. Since $\Delta \le \Theta$ and $\Delta = \sum(1 - 1/m_i)\Delta_i$ has only standard coefficients, so is (S, Δ). Therefore

3.3. **We may assume that (S, Δ) is fixed.** Let Y' be the normalization of a dominant component of $Y \times_X X'$. Consider the commutative diagram

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ \phi\ } & Y \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
X' & \xrightarrow{\ \psi\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Z' & \xrightarrow{\ \pi\ } & Z
\end{array}
\tag{1}
$$

where $X' \to Z' \to Z$ is the Stein factorization. It is clear that $\pi^{-1}(o)$ is a point (because $(f \circ \psi)^{-1}(o)$ is connected). We claim that $g'\colon Y' \to X'$ is a plt blow-up. It is clear that $\phi\colon Y' \to Y$ is a finite Galois morphism and its ramification divisor can be supported only in S. Denote $S' := \phi^{-1}(S)$. Then S' is the exceptional divisor of the blow-up $g'\colon Y' \to X'$. We have that

$$
K_{Y'} + S' = \phi^*(K_Y + S)
\tag{2}
$$

and this divisor is plt [15, 2.2], [8, 20.3]. This gives us that S' is normal (see 1.3). On the other hand, $-(K_{Y'} + S' + B') = -\phi^*(K_Y + S + B)$ if nef and big over Z'. By Connectedness Lemma 1.6 S' is connected near each fiber of $f' \circ g'$ (and hence it is irreducible). Thus we have that $g'\colon Y' \to X'$ is a plt blow-up. This shows also that $K_{X'} + f'_* D'$ is exceptional. Similar to (2) we have

$$
K_{Y'} + S' + D' = \phi^*(K_Y + S + D)
\tag{3}
$$

and again this divisor is plt near S' [15, 2.2], [8, 20.3]. Put $\Theta' := \mathrm{Diff}_{S'}(\Theta)$ and $\Delta' := \mathrm{Diff}_{S'}$. Thus we have a finite Galois morphism

$\varphi = \phi|_{S'} \colon S' \to S$ such that

$$K_{S'} + \Delta' = \varphi^*(K_S + \Delta) \quad \text{and} \quad K_{S'} + \Theta' = \varphi^*(K_S + \Theta). \quad (4)$$

3.4. **We may assume that** (S', Δ') **is fixed.** Indeed, by [15, Sect. 2] (see also [8, 20.3]) taking into account (4) we have $discr(S', \Theta') \geq discr(S, \Theta)$. As in 3.3 by [2] the set $\{(S', \Delta')\}$ is bounded.

Lemma 3.5. $\deg \varphi$ *is bounded.*

Proof. First we assume that $g(S)$ is a point. In this case both (S, Δ) and (S', Δ') are log del Pezzo surfaces. We have $0 < (K_{S'} + \Delta')^2 = (\deg \varphi)(K_S + \Delta)^2$. Since for $(K_{S'} + \Delta')^2$ and $(K_S + \Delta)^2$ there are only a finite number of possibilities, $\deg \varphi$ is bounded. In the case when $g(S)$ is a curve (S, Ξ) is a log del Pezzo for $\Xi = \mathrm{Diff}_S(B^\beta) \geq \Delta$ (see the proof of Lemma 2.5).

Lemma 3.6. *Let* (S, Δ) *be a projective log surface. Consider the set of boundaries on* S

$$\mathcal{M}(S, \Delta) := \{\Xi : \Xi \geq \Delta, \ K_S + \Xi \ \text{is klt and} \ -(K_S + \Xi) \text{is nef}\}.$$

Assume that $\mathcal{M}(S, \Delta) \neq \emptyset$. *Then there exists a boundary* $\Xi_{\max}$ *such that* $\Xi_{\max} \geq \Delta$, $-(K_S + \Xi_{\max})$ *is nef,* $K_S + \Xi_{\max}$ *is lc, and* $(K_S + \Xi_{\max})^2 \geq (K_S + \Xi)^2$ *for any* $\Xi \in \mathcal{M}(S, \Delta)$.

Note that $\Xi_{\max}$ is contained in the closure $\overline{\mathcal{M}(S, \Delta)}$.

Proof. We may assume that $-(K_S + \Xi)$ is big, otherwise $(K_S + \Xi)^2 = 0$ for all $\Xi \in \mathcal{M}(S, \Delta)$. Let $\Xi = \sum \xi_j \Xi_j \in \mathcal{M}(S, \Delta)$ and let Ξ_i be a component of Ξ such that $\Xi_i^2 \geq 0$. Then we have $\Xi - \alpha \Xi_i \in \mathcal{M}(S, \Delta)$ if $\Xi - \alpha \Xi_i \geq \Delta$. Indeed, if $(K_S + \Xi - \alpha \Xi_i) \cdot L > 0$ for some irreducible curve L, then $\Xi_i \cdot L < 0$, a contradiction. Further,

$$(K_S + \Xi - \alpha \Xi_i)^2 =$$
$$(K_S + \Xi)^2 - \alpha(K_S + \Xi) \cdot \Xi_i - \alpha(K_S + \Xi - \alpha \Xi_i) \cdot \Xi_i \geq (K_S + \Xi)^2.$$

So we may assume that for any component Ξ_i we have $\xi_i = 0$ whenever $\Xi_i^2 \geq 0$ and $\Xi_i \not\subset \mathrm{Supp}(\Delta)$. In particular, all components of Ξ are contained in $\mathrm{Supp}(\Delta)$ or have negative self-intersection number. It is easy to show that $NE(S)$ is polyhedral (see e. g. [16, 2.5], [12, 4.12]). Hence there is only a finite number of curves with negative self-intersections. Thus we may assume that components of Ξ belong to a finite set of curves. Finally, we find the maximum of the quadratic form $(K_S + \Xi)^2$ on the compact set $\overline{\mathcal{M}(S, \Delta)} \cap (\oplus \mathbb{R} \cdot [\Xi_i])$. Obviously, this maximal Ξ_{max} is rational. $\qquad\square$

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To finish the proof of Lemma 3.5 we take a boundary $\Xi_{\max}$ on S such as in 3.6 and define a boundary Ξ'_m on S' by the formula $K_{S'} + \Xi'_m = \varphi^*(K_S + \Xi_{\max})$ (see [15, 2.1], [8, 20.2]). It is clear that $\Xi'_m \geq \Delta'$ (because $\Xi_{\max} \geq \Delta$). Again we have

$$\mathrm{Const}_2 \geq (K_{S'} + \Xi'_m)^2 = (\deg \varphi)(K_S + \Xi_{\max})^2 \geq (\deg \varphi)\,\mathrm{Const}_1 > 0$$

This gives us $\deg \varphi \leq \mathrm{Const}_2 / \mathrm{Const}_1$ and proves Lemma 3.5. $\qquad\square$

Now we finish the proof of Proposition 3.2. Let G be the Galois group of $\psi\colon X' \to X$. Then G acts on Y' and $Y = Y'/G$. Moreover, S' is G-invariant, so we have the action of G on S'. Let $G_0 \subset G$ be a subgroup which acts on S' trivially. It is clear that $G_0 \subset G$ is a normal subgroup of index $[G : G_0] = \deg \varphi$. Let $G_1 := G/G_0$. Then $S = S'/G_1$. We have proved that the order of G_1 is bounded. It remains to show that the order of G_0 is also bounded. Denote it by r. Let us write

$$K_Y = g^*K_X + aS, \qquad K_{Y'} = g'^*K_{X'} + a'S'. \qquad (5)$$

By assumption $a > -1 + \varepsilon$. On the other hand, $\phi^*S = rS'$ and by the Hurwitz formula

$$K_{Y'} = \phi^*K_Y + (r-1)S', \qquad K_{X'} = \psi^*K_X. \qquad (6)$$

Combining (5) and (6) it is easy to obtain (cf. [15, §2], [7, 3.16])

$$a' + 1 = r(a+1) > r\varepsilon \qquad (7)$$

Let us consider a sufficiently general curve H on S'. If $g'(S')$ is a point, then we can take $H \in |-n_0(K_{S'} + \Delta')|$, where n_0 depends only on (S', Δ'). If $g'(S')$ is a curve, then we can take H as the general fiber of $S' \to g'(S')$. From (5) we have

$$K_{S'} + \Delta' \equiv (K_{Y'} + S')|_{S'} \equiv (1 + a')S'|_{S'}.$$

This gives us

$$H \cdot (K_{S'} + \Delta') = H \cdot (K_{Y'} + S') = (1 + a')H \cdot S', \qquad (8)$$

where the left side is equal to $-n_0(K_{S'} + \Delta')^2$ if $g'(S')$ is a point and $-2 + H \cdot \Delta' \geq -2$ if $g'(S')$ is a curve. In particular, $H \cdot (K_{S'} + \Delta')$ depends only on (S', Δ'), but not on Y'. Recall that the coefficients of $\Delta' = \mathrm{Diff}_{S'}$ are standard (see [15, 3.9], [8, 16.6]), so we can write $\Delta' = \sum_{i=1}^{r}(1 - 1/m_i)\Delta'_i$ where $m_i \in \mathbb{N}$, $r \geq 0$. Put $m' := \mathrm{l.c.m.}(m_1, \ldots, m_r)$. Again by [15, 3.9] both $m'S'$ and $m'(K_{S'} + \Delta')$ are Cartier along H. So we can rewrite (8) as $N = (1 + a')k$, where $N = -m'H \cdot (K_{S'} + \Delta')$ is a fixed natural number and $k = -m'(H \cdot S)$ is also natural. Thus by

(7) $N = (1 + a')k > kr\varepsilon \geq r\varepsilon$. This gives us that $r < N/\varepsilon$ is bounded and proves the proposition. $\square$

Remark (Suggested by the referee). On can prove Proposition 3.2 by working on the normalization of $\overline{Y}$ in the function field of Y'. In this way Lemma 3.5 becomes immediate from [2] and the case division by $\dim g(S)$ is not needed.

Proof of Theorem A. Let $f: X \to Z$ be a contraction from Theorem A. So, f is a log conic bundle or a log del Pezzo fibration. Assume that K_X has no regular non-klt complements. Then f is exceptional by 2.18 and it is sufficient to prove that singularities $(Z \ni o)$ (resp. multiplicities of $f^{-1}(o)$) are bounded in terms of ε.

First, consider the case $\dim Z = 2$. It is known that $(Z \ni o)$ is a quotient singularity, i. e. $(Z, o) = (Z', o')/G$, where $(Z' \ni o')$ is a non-singular germ and G is a finite group acting on Z' free outside o. We may assume that $G \subset GL_2(\mathbb{C})$. Two conjugate subgroups give us analytic isomorphic singularities $(Z \ni o)$. So it is sufficient to show boundedness of $|G|$. Let us construct the base change

$$
\begin{array}{ccc}
X' & \xrightarrow{\ \psi\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Z' & \xrightarrow{\ \pi\ } & Z
\end{array}
\tag{9}
$$

where X' is the normalization of the dominant component of $X \times_Z Z'$. It is clear that $\psi: X' \to X$ is a finite Galois morphism which is étale in codimension 1. Take a boundary F as in 2.2. If $K_X + F$ is plt, then the assertion follows by 3.1. Moreover, in our case there exists a regular complement of K_X (because $\lfloor F \rfloor$ is non-compact). If $K_X + F$ is not plt, then the degree of ψ is bounded by Proposition 3.2.

In the case $\dim Z = 1$ denote $L := f^{-1}(o)_{\mathrm{red}}$. Then $mL \sim 0$, where m is the multiplicity of $f^{-1}(o)$. This gives us the m-cyclic cover $\psi: X' \to X$ which is étale in codimension one. The rest is similar to the case above. $\square$

4. Examples and concluding remarks

Proposition 4.1 (cf. [9, 2.7]). *Let $f: X \to Z$ be a contraction as in Theorem A. Assume f is exceptional. Then there exists a divisor S of $K(X)$ such that $a(S, F) = -1$ for every non-klt complement $K_X + F$.*

Proof. Let us take any complement $K_X + F$ which is not klt. Then there is a (not necessarily exceptional) divisor S of the field $K(X)$ with discrepancy $a(S, F) = -1$. Let us take another non-klt complement

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$K_X + F^1$. We claim that $a(S, F^1) = -1$. Indeed, we can construct a continuous family of boundaries $F(t)$, $t \in [0, 1]$ such that $F(0) = F$, $F(1) = F^1$, $K_X + F(t)$ is lc, non-klt and numerically trivial for all $t \in [0, 1]$. To see this take an effective divisor L on Z containing o and put $F(t) = (1 - t)F + tF^1 + c(t)f^*L$, where $c(t)$ is the log canonical threshold of $(X, (1 - t)F + tF^1)$ with respect to f^*L (see [15], [7]). Since $c(t)$ can be computed from a finite number of linear inequalities (see [7, 8.5]), $c(t)$ is a picewise linear function in t. In particular, it is continuous. Obviously, $K_X + F(t) \equiv 0$. Further, for any divisor E of $K(X)$ the discrepancy $a(E, F(t))$ is also continuous in t. Put

$$t_0 := \sup\{t \in [0, 1] : a(S, F(t)) = -1\}.$$

By continuity, there is another divisor E such that $a(E, F(t_0)) = -1$. This contradicts to Corollary 2.18. $\qquad\square$

Below we consider explicit examples of contractions such as in Theorem A and we shall show that both cases (i) and (ii) of Theorem A can really occur.

Example 4.2. Let $(Z \ni o)$ be a two-dimensional klt singularity, let $X := \mathbb{P}^1 \times Z$, and let $f \colon X \to Z$ be the projection. Then f is a generically conic bundle as in Theorem A. It is known that K_Z is 1, 2, 3, 4 or 6-complementary in cases $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, respectively [15, 5.2.3]. Let $K_Z + \Delta$ be this n-complement with minimal n and let S, S_1 be two distinct sections of f. Then $K_X + f^*\Delta + S + S_1$ is an n-complement on X. This n is also minimal. Indeed, let $K_X + B$ be any m-complement on X. Take a general point P on the central fiber of f and consider the general section $H \ni P$ of f. Then $K_H + \mathrm{Diff}_H(f^*\Delta)$ is an m-complement on H near P. Since $(H \ni P) \simeq (Z \ni o)$, we have that K_Z is also m-complementary.

Example 4.3. Let $G \subset GL_2(\mathbb{C})$ be a finite subgroup without quasire-flections. Consider the natural projection $f \colon (\mathbb{C}^2 \times \mathbb{P}^1)/G \to \mathbb{C}^2/G$, where the action of G on $\mathbb{P}^1$ is induced from $\mathbb{C}^2$. Then f satisfies conditions of Theorem A. As in the example above one can show that there exists a regular complement. Similarly, the base surface here can have any lt singularity.

The examples above shows that the condition of ε-lt cannot be removed from Theorem A.

In the case $\dim Z = 1$ we can construct a huge number of examples as products $F \times \mathbb{C}^1 \to \mathbb{C}^1$, where F is a del Pezzo with klt singularities. If we take F so that K_F has no regular complements, we get an exceptional contraction.

The case of birational contractions such as in 2.1 was studied by Shokurov [15, Sect. 7]. Here we note only that the same arguments as in Sect. 3 allow us to show that for divisorial contractions $f\colon X \to Z \ni o$, where X has only ε-lt singularities one has either K_X has a regular complement or the index of $(Z \ni o)$ is bounded by $\mathcal{C}(\varepsilon)$.

Almost all arguments can be applied also to the study of log Fano varieties (i. e. the case when Z is a point). The problem here is to construct a boundary F as in 2.1. Then we have to find a very singular element in $|-nK_X|$ for some $n \in \mathbb{N}$ (so called "tiger"). Such a divisor can be constructed by using Riemann-Roch if $-K_X^3$ is sufficiently large. In the terminal case we can use also Riemann-Roch for Weil divisors (see [5], [1]).

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